

# A Non-vanishing Property for the Signature of a Path

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## Abstract

We prove that a continuous path with finite length in a real Banach space cannot have infinitely many zero components in its signature unless it is tree-like. In particular, this allows us to strengthen a limit theorem for signature recently proved by Chang, Lyons and Ni. What lies at the heart of our proof is a complexification idea together with deep results from holomorphic polynomial approximations in the theory of several complex variables.

## 1 Introduction and main result

In the seminal work of Hambly and Lyons [6] in 2010, it was shown that the *signature* of a continuous path  $w : [0, T] \rightarrow \mathbb{R}^d$  with finite length, which is the collection

$$\left\{ \int_{0 < t_1 < \dots < t_n < T} dw_{t_1} \otimes \dots \otimes dw_{t_n} : n \geq 1 \right\}$$

of global iterated integrals of all orders, uniquely determines the path  $w$  up to a tree-like equivalence (heuristically, a path is tree-like if it goes out and reverses back along itself). In particular, there is a unique *tree-reduced* path (i.e. not containing any tree-like pieces) up to reparametrization with minimal length in each tree-like equivalence class with given signature. Since then, it has been

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conjectured that the length  $L$  of a tree-reduced path  $w$  can be recovered from (the tail asymptotics of) its signature:

$$L = \limsup_{n \rightarrow \infty} \left\| n! \int_{0 < t_1 < \dots < t_n < T} dw_{t_1} \otimes \dots \otimes dw_{t_n} \right\|_{\text{proj}}^{\frac{1}{n}}, \quad (1.1)$$

where  $\|\cdot\|_{\text{proj}}$  is the projective tensor norm on the tensor product. This conjecture was proved by Hambly-Lyons [6] for  $C^1$ -paths (a stronger asymptotic result was obtained in this case) and piecewise linear paths, and it remains open for general bounded variation paths.

In a recent work of Chang, Lyons and Ni [3] (see also [4]), under reasonable tensor algebra norms, it was shown that the right hand side of (1.1) is indeed a limit when  $n$  is taken over degrees at which the signature is nonzero. To be precise, let  $V$  be a real Banach space and  $V^{\otimes n}$  ( $n \geq 1$ ) be the algebraic tensor products. Recall from [3] that a sequence of tensor norms  $\|\cdot\|_{V^{\otimes n}}$  are called *reasonable tensor algebra norms* if

- (i)  $\|\xi \otimes \eta\|_{V^{\otimes(m+n)}} \leq \|\xi\|_{V^{\otimes m}} \cdot \|\eta\|_{V^{\otimes n}}$  for  $\xi \in V^{\otimes m}$ ,  $\eta \in V^{\otimes n}$ ;
- (ii)  $\|\phi \otimes \psi\| \leq \|\phi\| \cdot \|\psi\|$  for  $\phi \in (V^{\otimes m})^*$ ,  $\psi \in (V^{\otimes n})^*$ , where the norms are the induced dual norms;
- (iii)  $\|P^\sigma \xi\|_{V^{\otimes n}} = \|\xi\|_{V^{\otimes n}}$  for  $\xi \in V^{\otimes n}$  and  $\sigma$  being a permutation of order  $n$ , where  $P^\sigma$  is the induced permutation operator on  $n$ -tensors.

It can be shown (c.f. Diestel and Uhl [5]) that the inequalities in (i) and (ii) are automatically equalities. The completion of  $V^{\otimes n}$  under  $\|\cdot\|_{V^{\otimes n}}$  is denoted as  $(V^{\otimes n}, \|\cdot\|_{V^{\otimes n}})$ . Examples of reasonable tensor norms include the projective, injective and Hilbert-Schmidt tensor norms. Throughout the rest of this article, we will always fix a choice of reasonable tensor algebra norms. The main result of [3] can be stated as follows<sup>1</sup>.

**Theorem 1.** *Let  $w : [0, T] \rightarrow V$  be a continuous path with finite length, and let  $g = (1, g_1, g_2, \dots)$  be the signature of  $w$ , i.e.*

$$g_n \triangleq \int_{0 < t_1 < \dots < t_n < T} dw_{t_1} \otimes \dots \otimes dw_{t_n} \in V^{\otimes n}, \quad n \geq 1.$$

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<sup>1</sup>Indeed, in [3] the authors claimed the convergence as  $n \rightarrow \infty$  without further restrictions. However, a careful examination of the proof suggests that the convergence was only proved along degrees at which the signature is nonzero. This was corrected in the corrigendum [4] of [3]. Theorem 1 stated above is the corrected version.

Define  $N(g)$  to be the set of positive integers  $n$  such that  $g_n \neq 0$ . Then

$$\lim_{\substack{n \rightarrow \infty \\ n \in N(g)}} \|n!g_n\|_{V^{\otimes n}}^{\frac{1}{n}} = \sup_{n \geq 1} \|n!g_n\|_{V^{\otimes n}}^{\frac{1}{n}}.$$

*Remark 1.* The result holds for arbitrary weakly geometric rough paths, or more generally, for any group-like elements, since the proof relies only on the shuffle product formula (c.f. (2.1) below) of signature which is a purely algebraic property. But with the same factorial normalization, the result is only interesting in the bounded variation case.

On the other hand, in Theorem 1, it is a priori not clear whether the limit can be taken over the whole integer sequence, or equivalently, whether a continuous path with finite length can have infinitely many zero components in its signature. In the present article, we provide a definite answer to this question.

**Theorem 2.** *Let  $w : [0, T] \rightarrow V$  be a continuous path with finite length in some real Banach space  $V$ . Then the signature of  $w$  has infinitely many zero components if and only if  $w$  is tree-like.*

An immediate consequence of the above theorem is the following strengthened version of Chang, Lyons and Ni's result.

**Corollary 1.** *Let  $w : [0, T] \rightarrow V$  be a continuous path with finite length in some real Banach space  $V$  whose signature is  $g = (1, g_1, g_2, \dots)$ . Then we have*

$$\lim_{n \rightarrow \infty} \|n!g_n\|_{V^{\otimes n}}^{\frac{1}{n}} = \sup_{n \geq 1} \|n!g_n\|_{V^{\otimes n}}^{\frac{1}{n}}.$$

We point out that it is possible and easy to construct non-tree-like *rough* paths having vanishing signature along a subsequence of degrees, and this makes our result non-trivial and surprising. For instance, the signature of the 2-rough path  $\mathbf{w}_t = \exp(t[e_1, e_2])$  over  $V = \mathbb{R}^2$  vanishes identically along odd degrees. More generally, if  $l_t$  is a continuous bounded variation path in the space of degree  $n$  homogeneous Lie polynomials, then the signature of the  $n$ -rough path  $\mathbf{w}_t = \exp(l_t)$  vanishes identically along degrees which are not multiples of  $n$ . Therefore, Theorem 2 has to be a non-rough-path property, and the core of the argument, unlike the proof of Theorem 1, has to be analytic.

## 2 Proof of the Main Theorem

The sufficiency part of Theorem 2 follows directly from the uniqueness result of Hambly and Lyons. For the necessity part, our proof consists of two main ingredients. The first one, which is a purely algebraic property, is to identify more zeros in the signature from the given ones. The second one, which is the core of proof and relies crucially on the bounded variation assumption, is to show that the path cannot have “too many” zeros in its signature unless it is tree-like. The algebraic ingredient is relatively elementary while the analytic ingredient relies on a complexification argument and deep results from several complex variables.

### 2.1 The algebraic ingredient

To fix notation, for a given positive integer  $d$ , denote  $(d)$  as the set of positive integer multiples of  $d$ . The set of positive integers is denoted as  $\mathbb{Z}_+$ .

**Lemma 1.** *Let  $A$  be a non-empty subset of  $\mathbb{Z}_+$  which is closed under addition. If  $\mathbb{Z}_+ \setminus A$  contains infinitely many elements, then there exists a positive integer  $d \geq 2$ , such that  $A \subseteq (d)$ .*

*Proof.* This is a direct consequence of the characterization of numerical semi-groups (c.f. Rosales and García-Sánchez [8], Lemma 2.1). Since it is elementary, for completeness we provide an independent proof in the appendix.  $\square$

Now let  $g = (1, g_1, g_2, \dots)$  be a tensor series, i.e.  $g_n \in V^{\otimes n}$  for each  $n$ . Recall that  $g$  is *group-like* if it satisfies the following so-called *shuffle product formula*:

$$g_m \otimes g_n = \sum_{\sigma \in \mathcal{S}(m,n)} P^\sigma(g_{m+n}) \quad \forall m, n \geq 1, \quad (2.1)$$

where  $\mathcal{S}(m, n)$  is the subset of  $(m, n)$ -shuffles in the permutation group of order  $m + n$ . It is standard that the signature of a weakly geometric rough path (in particular, of a bounded variation path) is group-like. By applying Lemma 1 to the context of group-like elements, we obtain the following result which is the algebraic ingredient for the proof of Theorem 2.

**Lemma 2.** *Let  $g$  be a group-like element. If  $g$  vanishes along a subsequence of degrees, then there exists a positive integer  $d \geq 2$  such that  $g$  vanishes identically along degrees outside  $(d)$ .*

*Proof.* Let  $N(g) \subseteq \mathbb{Z}_+$  be the set of degrees along which  $g$  vanishes. The result is trivial if  $N(g) = \mathbb{Z}_+$ . Otherwise, suppose that  $A \triangleq \mathbb{Z}_+ \setminus N(g)$  is non-empty. Let  $i, j \in N(g)$ . Since  $g_i$  and  $g_j$  are both non-zero, according to the shuffle product formula (2.1) and the reasonability of tensor norms, we have

$$i!j!\|g_i\|_{V^{\otimes i}} \cdot \|g_j\|_{V^{\otimes j}} \leq (i+j)!\|g_{i+j}\|_{V^{\otimes(i+j)}},$$

and thus  $g_{i+j} \neq 0$ . Therefore,  $A$  is closed under addition. Since  $N(g)$  is an infinite set by assumption, we conclude from Lemma 1 that  $A \subseteq (d)$  for some  $d \geq 2$ . In other words,  $g$  vanishes identically along degrees outside  $(d)$ .  $\square$

## 2.2 The analytic ingredient

Note that Lemma 2 relies only on the group-like property of signatures. To complete the proof of Theorem 2, it remains to show that the signature of a bounded variation path cannot vanish identically outside  $(d)$  for some  $d \geq 2$  unless it is tree-like.

Let us first describe the underlying intuition. Suppose that  $w$  is a bounded variation path whose signature vanishes identically outside  $(d)$ . If we complexify our underlying space and take  $\lambda$  to be a  $d$ -th root of unity, then the two paths  $w$  and  $\lambda \cdot w$  have the same complex signature. Since these two paths are still quite different even modulo tree-like pieces, it is reasonable to expect that this could not happen unless  $w$  itself is tree-like. However, as we will see, this is not a simple consequence of the uniqueness result in [6], and indeed there is a very subtle issue in the complex situation which constitutes the main challenge for this part.

### 2.2.1 The real case

To illustrate the idea better, we first consider the case in which no complexification is needed, i.e. when  $d$  is an even integer. In this case, the assumption implies that odd degree components of the signature of  $w$  are identically zero. Theorem 2 then follows from a simple topological lemma below and the general uniqueness result of Boedihardjo, Geng, Lyons and Yang [2] over  $\mathbb{R}$ .

**Lemma 3.** *Let  $f : V \rightarrow V$  be a continuous bijection over a real Banach space  $V$  whose only fixed point is the origin and it preserves spheres centered at the origin. Let  $w$  be a continuous path in  $V$  starting at the origin. If  $w$  and  $f(w)$  are equal up to a reparametrization, then  $w$  must be the trivial path.*

*Proof.* Suppose on the contrary that  $w$  is non-trivial. Then there exists some  $t > 0$  such that  $w_t \neq 0$ . Let  $\varepsilon \triangleq \|w_t\|_V$  and define

$$\tau \triangleq \inf\{0 \leq s \leq t : \|w_s\|_V = \varepsilon\}.$$

Note that  $w|_{[0,\tau]}$  is contained in the open ball  $B_\varepsilon$ . Since  $\|f(w)_\tau\|_V = \|w_\tau\|_V = \varepsilon$  and  $f(w)_\tau \neq w_\tau$ , by continuity, there exists some  $\delta > 0$ , such that

$$f(w)([\tau - \delta, \tau]) \cap w([0, \tau]) = \emptyset.$$

Since  $f(w)$  and  $w$  differ by reparametrization, we know that a subset of  $f(w)|_{[0,\tau-\delta]}$  must coincide with  $w|_{[0,\tau]}$ . This is not possible since by the assumption on  $f$ , we know that  $f(w)|_{[0,\tau-\delta]}$  is contained in the open ball  $B_\varepsilon$  while  $w_\tau$  lies on the boundary. Therefore,  $w$  must be trivial.  $\square$

Now we can give the proof of Theorem 2 when  $d$  is even. Given a path  $w$ , its *signature path* is the path defined by

$$\mathbb{W}_t \triangleq \left( 1, w_t, \int_{0 < s_1 < s_2 < t} dw_{s_1} \otimes dw_{s_2}, \dots, \int_{0 < s_1 < \dots < s_n < t} dw_{s_1} \otimes \dots \otimes dw_{s_n}, \dots \right),$$

which lives in the infinite tensor algebra  $T((V)) \triangleq \prod_{n=0}^{\infty} V^{\otimes n}$ . For each  $N \geq 1$ , the *truncated signature path* of order  $N$  is the projection of  $\mathbb{W}_t$  onto the truncated tensor algebra  $T^{(N)}(V) \triangleq \bigoplus_{n=0}^N V^{\otimes n}$  of order  $N$ .

*Proof of Theorem 2 when  $d$  is even.* Suppose that  $w : [0, T] \rightarrow V$  is a continuous path starting at the origin with finite length, whose signature  $g$  vanishes identically along odd degree components. Let  $\bar{w}$  be the unique tree-reduced path (up to reparametrization) having the same signature as  $w$ , i.e. the one which does not contain any tree-like pieces or equivalently whose signature path is simple (c.f. [2], Theorem 1.1 and Lemma 4.6). From assumption, the two paths  $-\bar{w}$  and  $\bar{w}$  have the same signature. According to the uniqueness theorem for signature in [2], they are equal up to tree-like equivalence. But  $-\bar{w}$  is also tree-reduced since its signature path is also simple. Therefore,  $-\bar{w}$  and  $\bar{w}$  must be equal up to reparametrization. According to Lemma 3 applied to the map  $f$  defined by  $f(v) = -v$ , we conclude that  $\bar{w}$  must be trivial and equivalently  $w$  is tree-like.  $\square$

*Remark 2.* In the above argument, we have not used the bounded variation property in an essential way, and the theorem holds for paths with finite  $p$ -variation for  $1 \leq p < 2$  without changing the proof. The non-rough-path regularity is used in the way that if the first level path is trivial then the signature (or equivalently, the signature path) is trivial, which is not true for general rough paths.

### 2.2.2 The complex case

Now we consider the case when  $d$  is an odd integer. Unlike the other case, it is hard to construct a real isomorphism of  $V$  leaving the signature invariant, and the simplest way to have such invariance is multiplying by a  $d$ -th root of unity which is now a complex number. In this way, we need to complexify the underlying space and signature needs to be understood in the complex sense. A crucial point one needs to be aware of is that the complex signature is defined through iterated integrals with respect to the complex variables only but *not* with their conjugates.

This will lead to a substantial challenge in the complex case to make the previous real argument work. Indeed, the real uniqueness result for signature does not hold over  $\mathbb{C}$ ! Heuristically, being tree-like is a real property and there exists non-tree-like paths with trivial complex signature. For instance, according to Cauchy's theorem, any simple and closed path with finite length living inside a one dimensional complex subspace of  $\mathbb{C}^n$  has trivial complex signature while it needs not be tree-like. Therefore, the complex version of the uniqueness result requires major modification, which at this point is unclear and unknown. However, for our particular problem, we can still obtain inspirations from the main strategy in the proof of the real uniqueness result. Our argument in this part relies on ideas developed in [2] and deep results from several complex variables.

To start with, we first introduce some standard notions about complexification of real Banach spaces. Recall that the complexification of  $V$  is defined by  $V_{\mathbb{C}} \triangleq V \otimes_{\mathbb{R}} \mathbb{C}$ , which is isomorphic to  $V \oplus V$  equipped with a complex scalar multiplication in the canonical way. A natural choice of norms on  $V_{\mathbb{C}}$ , known as the *Taylor complexification norm*, is defined by

$$\|x + iy\|_T \triangleq \sup_{0 \leq t \leq 2\pi} \|x \cos t - y \sin t\|_V, \quad x, y \in V.$$

The Taylor complexification norm satisfies

$$\|x + iy\|_T = \|x - iy\|_T, \quad \|x\|_T = \|x\|_V, \quad x, y \in V.$$

We always endow  $V_{\mathbb{C}}$  with this norm and the (complex) tensor products  $V_{\mathbb{C}}^{\otimes n}$  with the injective tensor norm. Let  $j_n : V^{\otimes n} \rightarrow V_{\mathbb{C}}^{\otimes n} \cong (V^{\otimes n})_{\mathbb{C}}$  be the canonical embedding.

**Lemma 4.** *For each  $n \geq 1$ ,  $j_n$  is continuous with norm at most one, and thus extends continuously to the completion of the algebraic tensor product.*

*Proof.* According to van Zyl [10], Theorem 2.3, the injective tensor norm on  $V_{\mathbb{C}}^{\otimes n}$  coincides with the Taylor complexification norm induced from the injective norm

on  $V^{\otimes a^n}$ . In addition, it is known (c.f. [5], Chapter 8, Proposition 3) that injective tensor norm is the smallest among all reasonable tensor norms. Therefore, for any  $\xi \in V^{\otimes a^n}$ ,

$$\|j_n(\xi)\|_{V_{\mathbb{C}}^{\otimes a^n}} = \|j_n(\xi)\|_T = \|\xi\|_{\text{inj}} \leq \|\xi\|_{V^{\otimes a^n}}.$$

□

**Lemma 5.** *Let  $w : [0, T] \rightarrow V$  be a continuous path with finite length. For each  $n \geq 1$ , let  $g_n$  (respectively,  $g_n^{\mathbb{C}}$ ) be the  $n$ -th degree component of its signature when  $w$  is viewed as a path in  $V$  (respectively, in  $V_{\mathbb{C}}$ ). Then  $g_n^{\mathbb{C}} = j_n(g_n)$ .*

*Proof.* Let  $\xi_m$  be the discrete Riemann sum approximation of  $g_n$ . Then  $j_n(\xi_m)$  is the discrete Riemann sum of  $g_n^{\mathbb{C}}$ . The result then follows from the continuity of  $j_n$  stated in Lemma 4. □

*Remark 3.* Lemma 5 remains true for any arbitrary rough path and its complexification. In addition, we only need to be careful about complexification of norms in the infinite dimensional setting as the finite dimensional case is trivial in terms of norm comparison.

It is easy to see that the notion of group-like property carries through to the complex case, and the complex signature of a weakly geometric complex rough path (in particular, of a complex bounded variation path) is group-like.

From now on, we fix  $d \geq 3$  to be an odd integer and  $\lambda \triangleq e^{2\pi i/d}$  to be a  $d$ -th root of unity. To prove Theorem 2 in this case, let  $w : [0, T] \rightarrow V$  be a continuous path with finite length starting at the origin whose signature vanishes identically outside  $(d)$ . We assume on the contrary that  $w$  is not tree-like (equivalently it has non-trivial signature) and look for a contradiction. As before in the real case, we may assume without loss of generality that  $w$  is tree-reduced. It is apparent from assumption that the path  $z_t \triangleq \lambda \cdot w_t$  has the same *complex* signature as  $w_t$ . The main difficulty here is that  $z$  has different *real* signature from  $w$  (when regarding  $V_{\mathbb{C}} = V \oplus V$  as a real vector space) so that the real uniqueness result does not apply.

To explain the underlying idea, assume for the moment that  $\dim V < \infty$  and  $w$  is a simple and closed path. If  $w$  is non-trivial, it is not hard to construct a *real* continuous one form  $\phi = \sum_j \phi_j(x) dx^j$  over  $V$  supported inside some small neighbourhood  $B \subseteq V$  of  $q \in \text{Im}(w) \setminus \{0\}$ , such that  $\int_0^T \phi(dw) = 1$ . Since over  $V_{\mathbb{C}}$ ,  $B$  is entirely separated from  $\text{Im}(z)$ , by zero extension each  $\phi_j$  extends to a continuous function  $\tilde{\phi}_j$  over the compact subset  $K \triangleq \text{Im}(w) \cup \text{Im}(z) \subseteq V_{\mathbb{C}}$ , and



therefore  $\phi$  extends to a continuous one form  $\bar{\phi} = \sum_j \bar{\phi}_j(z) dz^j$  (not containing  $d\bar{z}^j$ !) over  $V_{\mathbb{C}}$ . In particular, from the construction we see that

$$\int_0^T \bar{\phi}(dw_t) = 1 \text{ and } \int_0^T \bar{\phi}(dz_t) = 0.$$

The key to reaching a contradiction is the possibility of approximating  $\bar{\phi}$  by *holomorphic* polynomial one forms (i.e. polynomial in the complex variables but not in their conjugates). This turns out to be a rather deep problem in the theory of holomorphic polynomial approximations in several complex variables, and it can only be achieved in some very special situations (fortunately, our situation is special enough). Once we are able to replace  $\bar{\phi}$  by a complex polynomial one form  $p$ , a contradiction is then immediate since the integral depends only on  $p$  and the complex signature according to the shuffle product formula. If  $V$  is infinite dimensional and  $w$  is a general tree-reduced path, one needs to work with truncated signature paths and apply finite dimensional reduction in a proper way similar to the strategy developed in [2] when proving the real uniqueness result.

We first review a few results on holomorphic polynomial approximations in several complex variables that will be needed for our problem.

**Definition 1.** The *polynomial convex hull* of a compact subset  $K \subseteq \mathbb{C}^n$  is defined as

$$\hat{K} \triangleq \left\{ z \in \mathbb{C}^n : |p(z)| \leq \sup_{w \in K} |p(w)| \text{ for all complex polynomials } p \right\}.$$

A compact subset  $K$  is called *polynomially convex* if  $\hat{K} = K$ .

Polynomial convexity is closely related to uniform polynomial approximations, as seen from the following result.

**Theorem 3** (c.f. Levenberg [7], Page 97, Corollary). *Let  $K$  be a polynomially convex compact subset of  $\mathbb{C}^n$  with zero 2-dimensional Hausdorff measure. Then every continuous function over  $K$  can be uniformly approximated by holomorphic polynomials.*

**Example 1.** (1) Every compact subset  $K$  of  $\mathbb{R}^n$ , where  $\mathbb{R}^n$  is viewed as the real part of  $\mathbb{C}^n$ , is polynomially convex. Therefore, any continuous function on the image of a bounded variation path in  $\mathbb{R}^n$  can be uniformly approximated by complex polynomials. This can also be seen by applying the more standard real polynomial approximation theorems (e.g. Bernstein's theorem) and regard a real polynomial as a complex polynomial in the natural way.

(2) Let  $K$  be the unit circle in  $\mathbb{C}^1$ . Then the polynomial convex hull of  $K$  is the unit disk. This partly explains why not every continuous function on  $S^1$  can be uniformly approximated by holomorphic polynomials which is consistent with Cauchy's theorem.

The following result, which gives a way of verifying polynomial convexity in some special situations, is crucial for us.

**Theorem 4** (c.f. Weinstock [11], Theorem 1). *Let  $A$  be a real  $n \times n$  matrix which does not have purely imaginary eigenvalues of modulus greater than one. Define  $M \triangleq (A + i)\mathbb{R}^n \subseteq \mathbb{C}^n$ . Then every compact subset of  $M \cup \mathbb{R}^n \subseteq \mathbb{C}^n$  is polynomially convex.*

We now return to our signature problem. Recall that  $d \geq 3$  is an odd integer and  $\lambda = e^{i\theta}$  ( $\theta \triangleq 2\pi/d$ ) is a  $d$ -th root of unity. For each  $N \geq 1$ , define the real and complex spaces

$$\tilde{T}_{\text{inj}}^{(N)}(V) \triangleq \bigoplus_{\substack{k=1 \\ d \nmid k}}^N V^{\otimes_{\text{inj}} k}, \quad \tilde{T}^{(N)}(V_{\mathbb{C}}) \triangleq \bigoplus_{\substack{k=1 \\ d \nmid k}}^N V_{\mathbb{C}}^{\otimes k},$$

respectively, where ‘‘inj’’ means injective tensor norm. Note that  $\tilde{T}_{\text{inj}}^{(N)}(V)$  is canonically embedded inside  $\tilde{T}^{(N)}(V_{\mathbb{C}})$  as its real part. Define the dilation operator  $\delta_{\lambda} : \tilde{T}^{(N)}(V_{\mathbb{C}}) \rightarrow \tilde{T}^{(N)}(V_{\mathbb{C}})$  in the usual way by  $V_{\mathbb{C}}^{\otimes k} \ni g_k \mapsto \lambda^k g_k$ . As will be clear soon, the reason we work in the space  $\tilde{T}^{(N)}(V_{\mathbb{C}})$  instead of the more traditional truncated tensor algebra  $T^{(N)}(V_{\mathbb{C}})$  is for technical convenience of applying Theorem 4 (c.f. Lemma 7 below).

**Lemma 6.** *Let  $E$  be a real vector space and let  $\{v_1^j + \cdots + v_{r_j}^j : 1 \leq j \leq n\}$  be a linearly independent set. Then one can choose  $v_{l_j}^j$  ( $1 \leq l_j \leq r_j$ ) for each  $j$ , such that  $\{v_{l_j}^j : 1 \leq j \leq n\}$  are linearly independent.*

*Proof.* We write  $v^j \triangleq v_1^j + \cdots + v_{r_j}^j$ . Then there exists at least one  $v_{l_1}^1$  such that  $\{v_{l_1}^1, v^2, \dots, v^n\}$  are linearly independent, for otherwise  $v^1$  will be linearly dependent on  $\{v^2, \dots, v^n\}$  which is a contradiction. Similarly, there exists at least one  $v_{l_2}^2$  such that  $\{v_{l_1}^1, v_{l_2}^2, v^3, \dots, v^n\}$  are linearly independent. Now one can proceed by induction.  $\square$

**Lemma 7.** Let  $L : \tilde{T}_{\text{inj}}^{(N)}(V) \rightarrow \mathbb{R}^n$  be a real surjective continuous linear map, and extend  $L$  to a complex continuous linear map  $\bar{L} : \tilde{T}^{(N)}(V_{\mathbb{C}}) \rightarrow \mathbb{C}^n$  in the canonical way by

$$\bar{L}(u + iv) \triangleq L(u) + iL(v).$$

Let  $M \triangleq \delta_{\lambda}(\tilde{T}_{\text{inj}}^{(N)}(V)) \subseteq \tilde{T}^{(N)}(V_{\mathbb{C}})$ . Then every compact subset of  $\bar{L}(M) \cup \mathbb{R}^n$  is polynomially convex.

*Proof.* Since  $L$  is surjective, there exist elements  $g_1, \dots, g_n \in \tilde{T}_{\text{inj}}^{(N)}(V)$  such that  $\{L(g_1), \dots, L(g_n)\}$  form a basis of  $\mathbb{R}^n$ . Since each  $g_j$  is a sum of homogeneous tensors, according to Lemma 6, we may choose some  $\xi_{l_j} \in V^{\otimes_{\text{inj}} l_j}$  ( $1 \leq j \leq n$ ) such that  $\{L(\xi_{l_1}), \dots, L(\xi_{l_n})\}$  form a basis of  $\mathbb{R}^n$ . In addition, observe that

$$\begin{aligned} M &= \text{Span}_{\mathbb{R}} \{ \lambda^k \cdot \xi_k : 1 \leq k \leq N, d \nmid k, \xi_k \in V^{\otimes_{\text{inj}} k} \} \\ &= \text{Span}_{\mathbb{R}} \{ (\cos k\theta + i \sin k\theta) \cdot \xi_k : 1 \leq k \leq N, d \nmid k, \xi_k \in V^{\otimes_{\text{inj}} k} \}. \end{aligned}$$

Since  $d$  is odd, we know that  $\sin k\theta \neq 0$  for all  $k$  not being a multiple of  $d$ . Therefore,

$$M = \text{Span}_{\mathbb{R}} \{ (\cot k\theta + i) \cdot \xi_k : 1 \leq k \leq N, d \nmid k, \xi_k \in V^{\otimes_{\text{inj}} k} \}.$$

It follows that  $\bar{L}(M)$  is an  $n$ -dimensional real subspace of  $\mathbb{C}^n$  with basis  $\{(\cot l_j\theta + i) \cdot L(\xi_{l_j}) : 1 \leq j \leq n\}$ . In particular, if we define a non-degenerate real linear transform  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $A(L(\xi_{l_j})) \triangleq \cot l_j\theta \cdot L(\xi_{l_j})$ , then  $A$  does not have purely imaginary eigenvalues (indeed, all eigenvalues of  $A$  are given by  $\{\cot l_j\theta : 1 \leq j \leq n\}$ ), and  $\bar{L}(M) = (A + i)\mathbb{R}^n$ . According to Theorem 4, we conclude that every compact subset of  $\bar{L}(M) \cup \mathbb{R}^n$  is polynomially convex.  $\square$

Now we are in a position to give the proof of Theorem 2 when  $d$  is odd. Recall that  $w : [0, T] \rightarrow V$  is a non-trivial tree-reduced path with finite length starting at the origin, whose signature vanishes identically along degrees outside  $(d)$ . Viewed as paths in  $V_{\mathbb{C}}$ , we know that  $w$  and  $z \triangleq \lambda \cdot w$  have the same complex signature.

*Proof of Theorem 2 when  $d$  is odd.* First of all, since  $w$  is non-trivial, let  $I \subseteq (0, T)$  be a compact interval such that  $0 \notin w(I)$ . Let  $\mathbb{W}$  and  $\mathbb{Z}$  be the complex signature paths of  $w$  and  $z$  respectively, which live in the infinite complex tensor algebra  $T((V_{\mathbb{C}})) \triangleq \prod_{n=0}^{\infty} V_{\mathbb{C}}^{\otimes n}$ . Since  $w$  is tree-reduced, we know that  $\mathbb{W}$  and  $\mathbb{Z}$  are both simple. Also they have the same starting and end points respectively. Observe that

$$\mathbb{W}(I) \cap \mathbb{Z}([0, T]) = \emptyset, \tag{2.2}$$

for otherwise if  $\mathbb{W}_t = \mathbb{Z}_s = \delta_\lambda(\mathbb{W}_s)$  for some  $t \in I$  and  $s \in [0, T]$ , then  $0 \neq w_t = \lambda w_s$  which is absurd. Fix four points  $s < s' < t' < t$  in  $I$ . It follows that

$$\mathbb{W}([s, s']) \cap \mathbb{W}([t', t]) = \emptyset \quad (2.3)$$

and

$$\mathbb{W}([s', t']) \cap (\mathbb{W}([0, s] \cup [t, T])) = \emptyset. \quad (2.4)$$

In addition, from the triangle inequality we know that

$$\left\| \int_{u < t_1 < \dots < t_n < v} dw_{t_1} \otimes \dots \otimes dw_{t_n} \right\|_{V_{\mathbb{C}}^{\otimes n}} \leq \frac{\|w\|_{1\text{-var};[u,v]}}{n!}, \quad \forall n \geq 1 \text{ and } u \leq v,$$

and the same is true for the path  $z_t$ . Therefore, when  $N$  is large, all of the separation properties (2.2), (2.3) and (2.4) are preserved if we consider the complex truncated signature paths  $W^N$  and  $Z^N$  in  $T^{(N)}(V_{\mathbb{C}}) \triangleq \bigoplus_{n=0}^N V_{\mathbb{C}}^{\otimes n}$ . Choose  $N = dm + 1$  with some large  $m$  for this purpose.

We claim that the projections of  $W^N$  and  $Z^N$  onto  $\tilde{T}^{(N)}(V_{\mathbb{C}})$ , denoted as  $\tilde{W}^N$  and  $\tilde{Z}^N$  respectively, preserve all the previous three separation properties. We only verify (2.3) as the other cases can be treated in the same way. Let  $u \in [s, s']$  and  $v \in [t', t]$ . Suppose on the contrary that  $\tilde{W}_u^N = \tilde{W}_v^N$ . Since  $W_u^N \neq W_v^N$ , we conclude that  $\pi_{pd}(W_u^N) \neq \pi_{pd}(W_v^N)$  for some  $1 \leq p \leq m$ , where  $\pi_{pd}$  denotes the projection onto the  $pd$ -th component. But we know that  $\pi_1(W_u^N) = \pi_1(W_v^N) \neq 0$  since  $\tilde{W}_u^N = \tilde{W}_v^N$  and  $u, v \in I$ . Therefore, using the cross-norm property we have

$$\begin{aligned} & \left\| \pi_{pd}(W_u^N) \otimes \pi_1(W_u^N) - \pi_{pd}(W_v^N) \otimes \pi_1(W_v^N) \right\|_{V_{\mathbb{C}}^{\otimes (pd+1)}} \\ &= \left\| (\pi_{pd}(W_u^N) - \pi_{pd}(W_v^N)) \otimes \pi_1(W_u^N) \right\|_{V_{\mathbb{C}}^{\otimes (pd+1)}} \\ &= \left\| \pi_{pd}(W_u^N) - \pi_{pd}(W_v^N) \right\|_{V_{\mathbb{C}}^{\otimes pd}} \cdot \left\| \pi_1(W_u^N) \right\|_{V_{\mathbb{C}}} \\ &\neq 0, \end{aligned}$$

which implies that

$$\pi_{pd}(W_u^N) \otimes \pi_1(W_u^N) \neq \pi_{pd}(W_v^N) \otimes \pi_1(W_v^N).$$

According to the shuffle product formula, we conclude that

$$\pi_{pd+1}(W_u^N) \neq \pi_{pd+1}(W_v^N),$$

which is a contradiction to the assumption  $\tilde{W}_u^N = \tilde{W}_v^N$  since  $d \nmid pd + 1$ . Therefore, (2.3) holds for the path  $\tilde{W}^N$ .

Next, since  $\tilde{W}^N$  lives in  $\tilde{T}_{\text{inj}}^{(N)}(V)$  (the real part of  $\tilde{T}^{(N)}(V_{\mathbb{C}})$ ), according to the Hahn-Banach theorem (c.f. [2], Lemma 4.5), there exists a real continuous linear map  $L_1 : \tilde{T}_{\text{inj}}^{(N)}(V) \rightarrow \mathbb{R}^{n_1}$  with some  $n_1$ , such that

$$L_1 \left( \tilde{W}^N([s, s']) \right) \cap L_1 \left( \tilde{W}^N([t', t]) \right) = \emptyset \quad (2.5)$$

and

$$L_1 \left( \tilde{W}^N([s', t']) \right) \cap L_1 \left( \tilde{W}^N([0, s] \cup [t, T]) \right) = \emptyset. \quad (2.6)$$

Also for the same reason, there exists a real continuous linear map  $f : V \rightarrow \mathbb{R}^{n_2}$  with some  $n_2$ , such that  $0 \notin f(w(I))$ . By taking images we may assume that  $L_1$  and  $f$  are both surjective. Set  $L_2 \triangleq f \circ \pi_1 : \tilde{T}_{\text{inj}}^{(N)}(V) \rightarrow \mathbb{R}^{n_2}$  where  $\pi_1$  is the canonical projection onto the first degree component. With  $n \triangleq n_1 + n_2$ , define

$$L \triangleq L_1 \oplus L_2 : \tilde{T}_{\text{inj}}^{(N)}(V) \rightarrow \mathbb{R}^n \cong \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$$

and extend  $L$  to a complex continuous linear map  $\bar{L} : \tilde{T}^{(N)}(V_{\mathbb{C}}) \rightarrow \mathbb{C}^n$  in the canonical way. It is apparent that the separation properties (2.5) and (2.6) are still true in the space  $\mathbb{R}^n$  with  $L_1$  replaced by  $L$ . Moreover, we claim that in the space  $\mathbb{C}^n$ , we also have

$$\bar{L} \left( \tilde{W}^N(I) \right) \cap \bar{L} \left( \tilde{Z}^N([0, T]) \right) = \emptyset. \quad (2.7)$$

Indeed, suppose on the contrary that  $\bar{L}(\tilde{W}_t^N) = \bar{L}(\tilde{Z}_s^N)$  for some  $t \in I$  and  $s \in [0, T]$ . By looking at the  $L_2$ -component, we see that

$$f(w_t) = \cos \theta \cdot f(w_s) + i \sin \theta \cdot f(w_s).$$

This implies that  $f(w_s) = 0$  and thus  $f(w_t) = 0$ , which is a contradiction to the construction of  $f$ .

Now take four open subsets  $U_1, U_2, V_1, V_2$  of  $\mathbb{R}^n$ , such that

$$\begin{aligned} L \left( \tilde{W}^N([s, s']) \right) &\subseteq U_1, \quad L \left( \tilde{W}^N([t', t]) \right) \subseteq U_2, \\ L \left( \tilde{W}^N([s', t']) \right) &\subseteq V_1, \quad L \left( \tilde{W}^N([0, s] \cup [t, T]) \right) \subseteq V_2, \end{aligned}$$

and

$$U_1 \cap U_2 = V_1 \cap V_2 = \emptyset.$$

Define  $F, G \in C_c^\infty(\mathbb{R}^n)$  to be such that

$$F = 0 \text{ on } V_2, \quad F = 1 \text{ on } V_1,$$

and

$$G = 0 \text{ on } U_1, \quad G = 1 \text{ on } U_2.$$

Consider the smooth one form  $\Phi \triangleq FdG$  over  $\mathbb{R}^n$ . From the construction, we have

$$\begin{aligned} & \int_0^T \Phi(d(L\tilde{W}^N)_u) \\ &= \left( \int_0^s + \int_s^{s'} + \int_{s'}^{t'} + \int_{t'}^t + \int_t^T \right) \Phi(d(L\tilde{W}^N)_u) \\ &= 0 + 0 + \left( G(\tilde{W}_{t'}^N) - G(\tilde{W}_{s'}^N) \right) + 0 + 0 \\ &= 1 - 0 \\ &= 1. \end{aligned}$$

Now regard  $\Phi = \sum_{j=1}^n \Phi_j(x) dx^j$  as a real continuous one form over  $\text{Im}(L\tilde{W}^N) \subseteq \mathbb{R}^n$  and extend it to a continuous one form  $\bar{\Phi} \triangleq \sum_{j=1}^n \Phi_j(z) dz^j$  over the compact set

$$K \triangleq \bar{L} \left( \tilde{W}^N([0, T]) \right) \cup \bar{L} \left( \tilde{Z}^N([0, T]) \right) \subseteq \mathbb{C}^n$$

by zero extension. This is legal because of the separation property (2.7) and by construction  $\Phi = 0$  on  $L(\tilde{W}^N([0, s] \cup [t, T]))$ . It follows that

$$\int_0^T \bar{\Phi}(d(\bar{L}\tilde{W}^N)_u) = 1 \text{ and } \int_0^T \bar{\Phi}(d(\bar{L}\tilde{Z}^N)_u) = 0. \quad (2.8)$$

From Lemma 7, we know that  $K$  is polynomially convex. In addition, since  $K$  is the union of images of bounded variation paths, it has zero 2-dimensional Hausdorff measure. According to Theorem 3, we know that  $\bar{\Phi}$  can be uniformly approximated over  $K$  by holomorphic polynomial one forms in  $\mathbb{C}^n$ . In particular, it follows from (2.8) that there exists a holomorphic polynomial one form  $P = \sum_{j=1}^n P_j(z) dz^j$ , such that

$$\int_0^T P \left( d(\bar{L}\tilde{W}^N)_u \right) \neq \int_0^T P \left( d(\bar{L}\tilde{Z}^N)_u \right). \quad (2.9)$$

On the other hand, it is not hard to see from the shuffle product formula that the complex signature of  $\bar{L}\tilde{W}^N$  as a bounded variation path over  $\mathbb{C}^n$  is a function

of the complex signature of  $w$  as a bounded variation path over  $V_{\mathbb{C}}$  (c.f. [2], Lemma 4.2 and Lemma 4.3 for the more general rough path case). The same is true for  $\bar{L}\tilde{Z}^N$ . Since  $w$  and  $z$  have the same complex signature, we conclude that  $\bar{L}\tilde{W}^N$  and  $\bar{L}\tilde{Z}^N$  have the same complex signature. But this leads to a contradiction with (2.9) since the integral is a function of the complex signature according to the shuffle product formula again.

Therefore, the path  $w$  has to be tree-like and the proof of Theorem 2 in the case when  $d$  is odd is complete. □

*Remark 4.* The above separation property by holomorphic polynomial one forms relies crucially on the feature that we are having a real path  $w$  and its complex rotation  $z = \lambda \cdot w$  (or more precisely, a real path  $\tilde{W}^N$  and its complex dilation  $\tilde{Z}^N = \delta_{\lambda}(\tilde{W}^N)$ ). A similar separation property for two general complex paths is highly non-trivial, and to our best knowledge this question is not fully understood in the literature. We expect that a proper understanding on this question will be an essential ingredient if one wants to investigate the uniqueness problem for signature over the complex field.

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## Appendix

In this section, for completeness we give an independent proof of Lemma 1.

*Proof of Lemma 1.* Fix some  $n \in A$ . Apparently  $n \geq 2$ , otherwise by the additivity assumption, we have  $A = \mathbb{Z}_+$  which is a contradiction.

For each  $i \in \mathbb{Z}_n$  (the integer group modulo  $n$ ), denote

$$[i] \triangleq \{i, i + n, i + 2n, i + 3n, \dots\}.$$

Let  $G \subseteq \mathbb{Z}_n$  be the collection of elements  $i$  such that  $[i] \cap A \neq \emptyset$ . We claim that  $G$  is a subgroup of  $\mathbb{Z}_n$ . Indeed, for  $i, j \in G$ , if both of  $i + kn$  and  $j + ln$  belong to  $A$  for some  $k$  and  $l$ , by assumption we see that  $i + j + (k + l)n$  belongs to  $A$ . Thus  $G$  is closed under addition. Moreover, by the same reason, the inverse of  $i \in G$ , which is the congruence class of  $(n - 1)i$ , also belongs to  $G$ . Therefore,  $G$  is a subgroup of  $\mathbb{Z}_n$ .

Note that  $\mathbb{N} = \cup_{i \in \mathbb{Z}_n} [i]$  and  $\mathbb{N} \setminus A$  is an infinite set, so there must exist some  $i_0 \in \mathbb{Z}_n$  such that  $[i_0] \cap (\mathbb{N} \setminus A)$  is an infinite set. However, since  $n \in A$ , if  $i_0 + kn \in \mathbb{N} \setminus A$ , by the additivity assumption we see that  $i_0 + (k - 1)n \in \mathbb{N} \setminus A$ . Therefore, we conclude that  $[i_0] \subseteq \mathbb{N} \setminus A$  and thus  $G$  is a proper subgroup of  $\mathbb{Z}_n$ .

Since  $\mathbb{Z}_n$  is cyclic, as a subgroup  $G$  must also be cyclic. Let  $d \triangleq \min\{i : i \in G\} \geq 2$ . Then  $d$  is a generator of  $G$  in  $\mathbb{Z}_n$  and  $d$  is a common divisor of all elements in  $G$ . Note that  $d$  divides  $n$  by Language's theorem. Therefore, we conclude that

$$A \subseteq \bigcup_{i \in G} [i] \subseteq (d).$$

□

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